Junctions of anyonic Luttinger wires

Brando Bellazzini,[1] Pasquale Calabrese,[2] and Mihail Mintchev[2]
[1]*Scuola Normale Superiore and INFN, 56126 Pisa, Italy*
[2]*Dipartimento di Fisica, Università di Pisa and INFN, 56127 Pisa, Italy*
(Received 27 August 2008; revised manuscript received 10 December 2008; published 24 February 2009)

We present an extended study of anyonic Luttinger liquids wires jointing at a single point. The model on the full line is solved with bosonization and the junction of an arbitrary number of wires is treated imposing boundary conditions that preserve exact solvability in the bosonic language. This allows us to reach, in the low-momentum regime, some of the critical fixed points found with the electronic boundary conditions. The stability of all the fixed points is discussed.

DOI: 10.1103/PhysRevB.79.085122 PACS number(s): 73.63.Nm, 73.23.−b, 71.10.Pm

I. INTRODUCTION

In the last few years there has been a boom in the study of transport properties at the junction of multiple quantum wires.[1–26] This interest is largely motivated by the fact that junctions of three or more wires would naturally appear in any quantum circuit. Different frameworks have been developed to tackle this complicated problem that shows a rich phase diagram. In fact, despite of the universality in the bulk of the wires that are described by a Luttinger liquid,[27] different conditions at the junctions can lead to exotic phase diagrams (as, e.g., those in Refs. 1, 9, and 23) whose degree of universality is not yet understood. According to the renormalization-group (RG) theory of critical phenomena, the low-energy properties of a gapless system are captured by the stable fixed point of the RG flow, independently of microscopic (nonuniversal) details of the real system. In view of the universality it is worthy to investigate very simple models, even exactly solvable, that can have (because of symmetry reasons) the same fixed points of the real systems. For bulk one-dimensional (1D) models and in the case of a single boundary, conformal field theory provides a complete classification of the universality classes (see, e.g., Ref. 28), whose analogous for junctions (or star graph) is not yet known. For all these reasons, we investigate in this paper the Tomonaga-Luttinger (TL) model on a junction with an arbitrary number n of arms as depicted in Fig. 1 (a junction with two wires $n=2$ can be seen as a defect on the line, a problem that has been largely investigated[29–40] in the past). To solve this problem, at the junction we impose conditions that are probably not obvious for an electronic problem, but they show the advantage to be exactly solvable. The natural hope is that the electronic model, at least for some values of the couplings, would be in the domain of attraction of the fixed points found here.

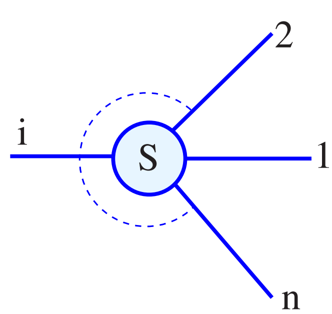

FIG. 1. (Color online) A quantum junction of n wires connected via a scattering matrix $\mathbb{S}$.

Furthermore we calculate the transport for particles with generalized anyonic statistics.[41] The reason for this generalization is twofold. On one hand the study of 1D anyonic model is attracting a renewed interest,[42–59] mainly motivated by possible experiments with cold atoms.[60] On the other hand, the transport of wires joined with a quantum Hall island is driven by anyonic excitations.[12] Also in this case we can wonder whether the different problems have some common fixed points. In 1D, anyonic statistics are described in terms of fields that at different points $(x_1 \neq x_2)$ satisfy the commutation relations

$$\Psi^\dagger(t,x_1)\Psi(t,x_2) = e^{-i\pi\kappa\epsilon(x_{12})}\Psi(t,x_2)\Psi^\dagger(t,x_1),$$

$$\Psi^\dagger(t,x_1)\Psi^\dagger(t,x_2) = e^{i\pi\kappa\epsilon(x_{12})}\Psi^\dagger(t,x_2)\Psi^\dagger(t,x_1), \qquad (1)$$

where $\epsilon(x)$ is the sign function $[\epsilon(z)=-\epsilon(-z)=1$ for $z>0$ and $\epsilon(0)=0]$ and $x_{12}=x_1-x_2$. κ is called statistical parameter and equals 0 for bosons and 1 for fermions. Other values of κ give rise to general anyonic statistics "interpolating" between the two familiar ones.

The TL model emerges naturally in the description of spinless fermions in 1D (and so electrons when the spin degrees of freedom are not important, but spin is also easily introduced in the formalism). In fact, starting from fermions hopping on a chain, linearizing the dispersion relation close to the Fermi surface at $\pm k_F$ and taking the continuum limit, one arrives to the standard TL Hamiltonian[27]

$$\mathcal{H} = \int dx[v_F(\psi_1^* i\partial_x\psi_1 - \psi_2^* i\partial_x\psi_2) + g_+\rho_+^2 + g_-\rho_-^2], \qquad (2)$$

where $\psi_{1,2}(t,x)$ are the two complex fields representing free-fermions left and right movers, v_F is the Fermi velocity, i.e., the speed of the noninteracting fermions, and

$$\rho_\pm(t,x) = [\psi_1^*(t,x)\psi_1(t,x) \pm \psi_2^*(t,x)\psi_2(t,x)] \qquad (3)$$

are the two independent charge densities. All the interaction is encoded in the coupling constants $g_\pm$ [often the couplings $g_{2,4}=2(g_+\mp g_-)$ are used]. Eventual irrelevant coupling terms of degree greater than 4 have been dropped. For $g_+>g_-$ the model is repulsive and it is attractive in the opposite case.

A similar reasoning can be repeated for anyonic degrees of freedom and the Hamiltonian is always given by Eq. (2), but with ψ_α satisfying the commutation relations (1); ψ_1 with κ and ψ_2 with $-\kappa$. Thus, when $\kappa=1$ the model is the well-known fermionic TL model, while the bosonic limit $\kappa \to 0$ is not well defined in this formalism as will be clearer in the following. We stress that this anyonic model is different from the gases discussed elsewhere[42,43,46,49,56] that also have a Luttinger liquid description. As in the fermionic case, the model is naturally solved exactly through bosonization.[27]

This Hamiltonian defines completely the system on each wire. To complete the description of the junction such as the one shown in Fig. 1 we have to define the interaction between the n wires. From an electronic point of view it is natural to have a term of the form[1]

$$\psi^*_\alpha(t,0,i)\mathcal{B}_{\alpha\beta ij}\psi_\beta(t,0,j), \tag{4}$$

where $\alpha,\beta=1,2$ and $i,j=1\ldots n$. The matrix $\mathcal{B}$ defines the boundary interaction among the fields ψ. Although very natural, this boundary condition is quite complicated after bosonization because it involves exponential boundary interactions of the bosonic fields. As a consequence the theory with this interaction term is no longer exactly solvable with bosonization, and very smart and complicated methods must be employed to extract the low-energy behavior from it.[1,9] In this paper we take an alternative approach that is to modify the junction couplings in such a way to preserve the exact solvability after bosonization.[20–22] The main idea is to impose the boundary condition directly on the bosonic degrees of freedom trying to have the same symmetries as in the ψ counterpart. The two problems can obviously have a different structure of fixed points, but, as stressed above, the natural hope is that the junction defined by Eq. (4) shares some of the anyonic fixed points with the ones found here, as it is well known to happen for fermions. The clear advantage of our approach is that keeping exactly solvability, the results are obtained with a relative little effort, compared to analogous ones for Eq. (4).

The paper is organized as follows. In Sec. II we introduce the anyonic TL model and solve it on the full line. In Sec. III after introducing the general features of the junction and the importance of conservation laws, we first present the standard solution on half line and then generalize it to the generic junction. In Sec. IV we study the stability of the found fixed points following the RG flow. Finally in Sec. V we draw our conclusion and discuss issues that need further investigation. In Appendixes A and B we report the technicality of bosonization and the description of the fixed points of the junction.

II. ANYONIC TOMONAGA-LUTTINGER MODEL

As already mentioned, the main goal of this paper is to investigate the Tomonaga-Luttinger model on half infinite quantum wires jointing in a single junction. However, in order to fix the notation and some basic tools, it is instructive to sketch first the solution of model on the line. In doing that we will focus on the general anyonic solution, which contains the more familiar fermionic one as a special case. The model is defined by the Hamiltonian (2) in which the space variable x is integrated on the full real axis. The corresponding equations of motion are

$$i(\partial_t - v_F\partial_x)\psi_1(t,x) = 2g_+\rho_+(t,x)\psi_1(t,x) + 2g_-\rho_-(t,x)\psi_1(t,x),$$

$$i(\partial_t + v_F\partial_x)\psi_2(t,x) = 2g_+\rho_+(t,x)\psi_2(t,x) + 2g_-\rho_-(t,x)\psi_2(t,x). \tag{5}$$

Bosonization[27] is the basic tool to quantize and solve these equations of motion. In fact, the solution can be expressed in terms of the right- and left-moving scalar fields $\varphi_{R,L}$. The standard details of the solution can be found in textbooks[27] and are reported in Appendix A to make this paper self-contained. The method is based on the change of variable

$$\psi_1(t,x) \propto :e^{i\sqrt{\pi}[\sigma\varphi_R(vt-x)+\tau\varphi_L(vt+x)]}:, \tag{6}$$

$$\psi_2(t,x) \propto :e^{i\sqrt{\pi}[\tau\varphi_R(vt-x)+\sigma\varphi_L(vt+x)]}:, \tag{7}$$

where the proportionality constants are explicitly given in Appendix A and $:\cdots:$ denotes the normal product relative to the creation and annihilation operators of φ fields. σ, τ, and v are three real parameters to be determined inserting these expressions in the equations of motion. Without loss of generality we take $\sigma \geq 0$ and assume that

$$\sigma \neq \pm \tau. \tag{8}$$

The charge densities take the very simple form

$$\rho_\pm(t,x) = \frac{-1}{2\sqrt{\pi}(\tau \pm \sigma)}[(\partial\varphi_R)(vt-x) \pm (\partial\varphi_L)(vt+x)]. \tag{9}$$

Imposing the current conservation

$$\partial_t\rho_\pm(t,x) - v\partial_x j_\pm(t,x) = 0, \tag{10}$$

one gets the currents

$$j_\pm(t,x) = \frac{(\partial\varphi_R)(vt-x) \mp (\partial\varphi_L)(vt+x)}{2\sqrt{\pi}(\tau \pm \sigma)}. \tag{11}$$

Using the exchange properties of φ, one can easily show that that the field ψ_1 satisfies the anyonic commutation relations given in Eq. (1) with statistical parameter

$$\kappa = \tau^2 - \sigma^2. \tag{12}$$

According to Eq. (8), $\kappa \neq 0$, which shows explicitly that the bosonic limit is not well defined in this context. The exchange relations of ψ_2 follow from Eq. (1) with the substitution $\kappa \mapsto -\kappa$, implying that ψ_α are both anyon fields, which become canonical fermions for $\kappa=1$.

The quantum equations of motion are obtained from Eq. (5) by replacing $\rho_\pm(t,x)\psi_\alpha(t,x) \mapsto :\rho_\pm\psi_\alpha:$ giving

$$\tau(v - v_F)\pi = \frac{g_+}{\tau+\sigma} + \frac{g_-}{\tau-\sigma}, \tag{13}$$

$$\sigma(v+v_F)\pi=\frac{g_+}{\tau+\sigma}-\frac{g_-}{\tau-\sigma}, \tag{14}$$

which, combined with Eq. (12), determine σ, τ, and the velocity v in terms of the coupling constants $g_\pm$ and the statistical parameter κ. In terms of the variables $\zeta_\pm=\tau\pm\sigma$, one obtains the system of equations

$$\zeta_+\zeta_-=\kappa, \tag{15}$$

$$v\zeta_+^2=v_F\kappa+\frac{2}{\pi}g_+, \tag{16}$$

$$v\zeta_-^2=v_F\kappa+\frac{2}{\pi}g_-, \tag{17}$$

with solution

$$\zeta_\pm^2=|\kappa|\left(\frac{\pi\kappa v_F+2g_+}{\pi\kappa v_F+2g_-}\right)^{\pm 1/2}, \tag{18}$$

$$v=\frac{\sqrt{(\pi\kappa v_F+2g_-)(\pi\kappa v_F+2g_+)}}{\pi|\kappa|}. \tag{19}$$

The relations (18) and (19) are the anyonic realization of the well-known result valid for canonical fermions in the TL model (the traditionally used parameter K in our notation coincides for $\kappa=1$ with $\zeta_-^2=\zeta_+^{-2}$; for comparison to Refs. 9 and 23 the notation is $g=K^{-1}$). The stability conditions of the model is $2g_\pm>-\pi\kappa v_F$, which ensures σ, τ, and v to be real and finite.

From the previously given mapping it is easy to write the Hamiltonian in terms of the bosonic fields obtaining

$$\mathcal{H}=\frac{v}{2}\int dx[(\partial_x\theta)^2+(\partial_x\varphi)^2], \tag{20}$$

where

$$\varphi(t,x)=\frac{1}{2}[\varphi_R(vt-x)+\varphi_L(vt+x)], \tag{21}$$

$$\theta(t,x)=\frac{1}{2}[\varphi_R(vt-x)-\varphi_L(vt+x)], \tag{22}$$

where θ is the so-called dual field. Notice that the Hamiltonian is slightly different from the usual one in the literature because we adsorb the coupling constant g (or K) in the definition of the fields.

It is worth commenting at this point the internal symmetries of the TL Hamiltonian because they will characterize the quantization on the junction. The TL Hamiltonian (2) is left invariant by the two independent $U(1)$ phase transformations usually denoted as $U(1)\otimes\widetilde{U}(1)$,

$$\psi_\alpha\to e^{is}\psi_\alpha,\quad \psi_\alpha^*\to e^{-is}\psi_\alpha^*, \tag{23}$$

$$\psi_\alpha\to e^{-i(-1)^\alpha\widetilde{s}}\psi_\alpha,\quad \psi_\alpha^*\to e^{i(-1)^\alpha\widetilde{s}}\psi_\alpha^*. \tag{24}$$

In the bosonic language they correspond to the independent shift invariance of the (compactified) fields $\varphi_{R,L}$. We will see that on the junction, the left and right movers are not independent anymore and the two $U(1)$ symmetries cannot be conserved simultaneously.

One of the main advantages of bosonization is that after having solved the equations of motion, it is straightforward to obtain all the correlation functions (also at finite temperature) just by commuting the fields φ in the exponential forms for ψ using Eq. (A15). In fact, in terms of the basic correlator

$$\mathcal{D}(x)=\frac{1}{\mathrm{i}(x-\mathrm{i}\epsilon)}, \tag{25}$$

the zero-temperature (Fock representation) field correlation functions are

$$\langle\psi_1^*(t_1,x_1)\psi_1(t_2,x_2)\rangle=\frac{1}{2\pi}[\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2},$$

$$\langle\psi_2^*(t_1,x_1)\psi_2(t_2,x_2)\rangle=\frac{1}{2\pi}[\mathcal{D}(vt_{12}-x_{12})]^{\tau^2}[\mathcal{D}(vt_{12}+x_{12})]^{\sigma^2}, \tag{26}$$

with $x_{12}=x_1-x_2$ and $t_{12}=t_1-t_2$. Scale invariance is manifest and one can read the dimension of ψ_α

$$d_{\mathrm{line}}=\frac{1}{2}(\sigma^2+\tau^2)=\frac{1}{4}(\zeta_+^2+\zeta_-^2). \tag{27}$$

All the other two-point field correlation functions vanish because of Eq. (8) and the neutrality condition $[U(1)\otimes\widetilde{U}(1)$ symmetry]. Analogously for the $U(1)$ density one finds

$$\langle\rho_+(t_1,x_1)\rho_+(t_2,x_2)\rangle=\frac{1}{(2\pi\zeta_+)^2}\{[\mathcal{D}(vt_{12}-x_{12})]^2+[\mathcal{D}(vt_{12}+x_{12})]^2\}, \tag{28}$$

and straightforwardly the ones for ρ_- and $j_\pm$ are obtained. We notice that all these correlation functions correctly agree with the general expression for a harmonic anyonic fluid[48] with only one harmonic term given by the Luttinger mode.

The generalization to finite temperature β^{-1} (Gibbs representation) is simply obtained with the replacement $\mathcal{D}(x)\to\mathcal{D}_\beta(x)$ with

$$\mathcal{D}_\beta(x)=\left[\frac{\mathrm{i}\beta}{\pi}\sinh\left(\frac{\pi x}{\beta}-\mathrm{i}\epsilon\right)\right]^{-1}, \tag{29}$$

and introducing the chemical potentials explicitly,

$$\langle\psi_1^*(t_1,x_1)\psi_1(t_2,x_2)\rangle_\beta=\frac{1}{2\pi}e^{i\mu_R\sigma(vt_{12}-x_{12})+i\mu_L\tau(vt_{12}+x_{12})}\times[\mathcal{D}_\beta(vt_{12}-x_{12})]^{\sigma^2}[\mathcal{D}_\beta(vt_{12}+x_{12})]^{\tau^2}, \tag{30}$$

and similarly for the other correlations. The right and left chemical potentials are

$$\mu_R = \frac{\mu}{\zeta_+} - \frac{\tilde{\mu}}{\zeta_-}, \quad \mu_L = \frac{\mu}{\zeta_+} + \frac{\tilde{\mu}}{\zeta_-}, \tag{31}$$

where μ and $\tilde{\mu}$ are the ones associated with the $U(1) \otimes \tilde{U}(1)$ charges.

III. JUNCTION OF TOMONAGA-LUTTINGER LIQUIDS

A. Boundary conditions and symmetries

After the previous preliminary considerations on the line, we investigate below the TL model at a junction such as the one shown in Fig. 1. In mathematical physics literature these junctions are usually called *star graphs* and they represent the building blocks for more general "quantum graph" networks (see for a review Ref. 61). We now fix all the notations on the junction that we call Γ. We indicate the jointing point of the junction as V. Each point P in the bulk $\Gamma \setminus V$ (i.e., of the wires) can be parametrized by the pair (x,i), where $i=1,\ldots,n$ labels the edge E_i and $x \in (0,\infty)$ is the distance of P from the vertex V along that edge. We stress that, as physically suggested, the embedding of Γ and the relative position of the edges in the "ambient space" are irrelevant.

The dynamics of each wire (edge) is still given by the Hamiltonian (2), but now $\psi_\alpha = \psi_\alpha(t,x,i)$ and $x>0$. As already discussed in Sec. I, in order to fix the solution one must impose some boundary conditions at the vertex V at $x=0$. The simplest boundary condition one can imagine is linear in ψ_α and is generated by the boundary term in Eq. (4) that makes the model nonexactly solvable for general couplings (see, e.g., Ref. 1 for free fermions and also Ref. 9 for infinite repulsive coupling).

An alternative which preserves the exact solvability after bosonization has been proposed.[20–22] The main idea is to impose the boundary condition directly on the bosonic degrees of freedom, selecting those of them which ensure unitary time evolution of the fields φ. This is guaranteed only if the boundary conditions are linear in the fields φ and its first derivatives. So we can parametrize these boundary conditions by a generic $n \times n$ unitary matrix $\mathbb{U}$,[20,21,62,63]

$$\sum_{j=1}^{n} [\lambda(\mathbb{I}-\mathbb{U})_{ij}\varphi(t,0,j) - \mathrm{i}(\mathbb{I}+\mathbb{U})_{ij}(\partial_x\varphi)(t,0,j)] = 0, \tag{32}$$

and $\lambda>0$ is a parameter with dimension of mass needed to recover the correct physical dimensions. Since bosonization expresses physical charges linearly in φ, we shall see below that these boundary conditions simply state how the charges are parceled out among the wires at the vertex.

The analysis of the fixed point is greatly simplified if we assume time-reversal invariance. This implies that the matrix $\mathbb{U}$ must be real, that together with unitarity leads to a symmetric matrix $\mathbb{U}$, i.e.,

$$\mathbb{U}^t = \mathbb{U}, \tag{33}$$

giving a further constraint on the possible boundary terms. A nontrivial magnetic flux (breaking time reversal) has been considered[9] and resulted in a more complicated fixed point structure. When dealing with anyon excitation, it would be more natural to consider non-time-reversal models because the magnetic field needed to produce the anyons breaks the symmetry. However this would complicate the analysis and in some regime it could be only an irrelevant perturbation. Thus in the following we will always assume time-reversal invariance and leave the study of the effect of its breaking to a future work.

The boundary condition (32) is equivalent[20,21] to an interaction with a pointlike defect localized at the vertex of the graph. The scattering matrix associated with this interaction is[20,21,62]

$$\mathbb{S}(k) = -[\lambda(\mathbb{I}-\mathbb{U}) + k(\mathbb{I}+\mathbb{U})]^{-1}[\lambda(\mathbb{I}-\mathbb{U}) - k(\mathbb{I}+\mathbb{U})] \tag{34}$$

and has transparent physical meaning: the diagonal element $\mathbb{S}_{ii}(k)$ represents the reflection amplitude on the edge E_i, whereas $\mathbb{S}_{ij}(k)$ with $i \neq j$ equals the transmission amplitude from E_i to E_j. Equation (34) makes also clear the meaning of the boundary terms λ and $\mathbb{U}$: for $\lambda \neq 0$ we have $\mathbb{S}(k=\lambda)=\mathbb{U}$, i.e., λ fixes the momentum scale at which the scattering matrix is given exactly by $\mathbb{U}$.

By construction the scattering matrix (34) is unitary

$$[\mathbb{S}(k)]^* = [\mathbb{S}(k)]^{-1} \tag{35}$$

and satisfies Hermitian analyticity

$$[\mathbb{S}(k)]^* = \mathbb{S}(-k). \tag{36}$$

Moreover, time-reversal invariance (33) implies

$$[\mathbb{S}(k)]^t = \mathbb{S}(k). \tag{37}$$

For simplicity we assume in this paper that $\mathbb{U}$ is such that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{ikx}\mathbb{S}_{ij}(k) = 0, \quad x>0, \tag{38}$$

which guarantees that $\mathbb{S}(k)$ has no bound states (see Ref. 64 for an extension to bound states).

The boundary conditions strongly influence the symmetry content on the junction. Each symmetry in the bulk gives a conserved charge Q [with density $\rho(x,t)$] because of the Noether theorem. If we want to keep the conservation of Q at the junction we *must* impose from the beginning that the currents $j(x,t)$ corresponding to the given density $\rho(x,t)$ are conserved at the vertex. This results in $\Sigma_{i=1}^{n} j(0,t)=0$ for all times. This is the *Kirchhoff's rule*, which must be imposed in the vertex in order to generate a time-independent charge from a given current. A basic example is given by the energy, which is a conserved quantity in the bulk. Because of unitarity, the matrix $\mathbb{U}$ in Eq. (32) parametrizes all boundary conditions which ensure the Kirchhoff rule for the energy-momentum tensor of φ and thus the time independence of the relative Hamiltonian. This means that there is no dissipation at the junction: if the energy flows out from one wire should flow in another one. We stress that the Kirchhoff's rule for gapless models on a graph is the generalization of the celebrated result that scale invariance implies holomorphic and antiholomorphic components of the energy tensor to be equal in boundary conformal field theory.[28,65]

Energy is not the only conserved quantity. In our formalism it is conserved by construction, but all other conservation laws we want to keep on the junction must be imposed by hand with appropriate Kirchhoff's rules. However it may happen that different conserved currents can generate contradictory Kirchhoff's rules, resulting in obstructions for lifting all symmetries on the line to symmetries on Γ.[66] In this case one can preserve on Γ one of the corresponding symmetries, but not all of them. This is actually the case for the $U(1) \otimes \widetilde{U}(1)$ symmetry of the TL model. In fact, the relative Kirchhoff rules generate[21] the following further constraints on $\mathbb{U}$:

$$\sum_{i=1}^{n} j_+(t,0,i) = 0 \Leftrightarrow \sum_{i=1}^{n} \mathbb{S}_{ji}(k) = \sum_{i=1}^{n} \mathbb{U}_{ji} = 1, \tag{39}$$

$$\sum_{i=1}^{n} j_-(t,0,i) = 0 \Leftrightarrow \sum_{i=1}^{n} \mathbb{S}_{ji}(k) = \sum_{i=1}^{n} \mathbb{U}_{ji} = -1, \tag{40}$$

which cannot be satisfied simultaneously. $U(1)$ is linked to the electric charge conservation and it is then natural to require the conservation of Eq. (39) while breaking Eq. (40). However also the opposite prescription has some interests. Notice that the duality transformation (A12) on Γ maps the matrix $\mathbb{U}$ [and so $\mathbb{S}(k,\lambda)$] in $-\mathbb{U}$ [$-\mathbb{S}(k^{-1},\lambda^{-1})$]. Consequently duality maps the vertex conservation of $U(1)$ in $\widetilde{U}(1)$.

The matrix conductance $\mathbb{G}$ of the junction can be obtained in linear-response theory. Since it involves only currents, the calculation is the same as for free bosons,[21,22] but with the renormalized current in Eq. (11), leading to an overall normalization

$$\mathbb{G} = \frac{1}{2\pi\zeta_+^2}(\mathbb{I}-\mathbb{S}) = G_{\text{line}}(\mathbb{I}-\mathbb{S}). \tag{41}$$

Thus the dependence of the conductance on the anyonic parameter is only through the renormalization constant ζ_+ in Eq. (18). Because of unitarity $|\mathbb{S}_{ii}| \leq 1$, we have

$$0 \leq \mathbb{G}_{ii} \leq 2G_{\text{line}}. \tag{42}$$

In the following, we will call conductance G the diagonal element $\mathbb{G}_{ii}$ in the case it does not depend on the wire index i.

It is worth mentioning that a similar approach (called delayed evaluation of boundary condition) working also with fermion boundary conditions was developed by Chamon and co-workers.[9,23] It basically amounts to leave in the half line, right and left movers unconstrained in the bulk, constructing then the tunneling operators, and only later choosing an R matrix (R for reflection, it can be easily rewritten as an S matrix) such that one of these processes pins the correct boundary conditions. In the appendix A of Ref. 23 the conductance is written in terms of an $n \times n$ R, which agrees with the results here and elsewhere.[21,22]

B. Half line

It is instructive to start with the well-known case $n=1$, namely, the half line, since some features of the generic junction are already manifest in this case. The matrices $\mathbb{U}$ and $\mathbb{S}$ are just numbers U and S. Setting $U=e^{-2i\alpha}$, we get

$$S(k) = \frac{k-i\eta}{k+i\eta}, \tag{43}$$

with

$$\eta = \lambda \tan(\alpha), \quad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}. \tag{44}$$

As expected the S matrix (43) corresponds to full reflection and describes the mixed (Robin) boundary condition

$$(\partial_x\varphi)(t,0) - \eta\varphi(t,0) = 0. \tag{45}$$

The condition (38) implies $\eta \geq 0$ or equivalently $0 \leq \alpha \leq \pi/2$. $\alpha=0$ and $\alpha=\pi/2$ correspond to Neumann and Dirichlet boundary conditions, respectively. These two points define the *only* bosonic scale-invariant boundary conditions on the half line. Instead of imposing the condition (45), we can add a term to the Hamiltonian in such a way to generate it as a further equation of motion. The resulting total Hamiltonian is

$$\mathcal{H}_{\text{tot}} = \mathcal{H} + \eta\varphi^2(t,0), \tag{46}$$

with $\mathcal{H}$ the bulk term given by Eq. (20), obviously defined only on the half line, i.e., the integral is over $x \in (0,\infty)$.

The main effect of the boundary in $x=0$ is to couple right and left movers by means of the boundary condition (45). In particular, at criticality, Eq. (45) implies that

$$\varphi_L(\xi) = \varphi_R(\xi), \quad \eta = 0, \tag{47}$$

$$\varphi_L(\xi) = -\varphi_R(\xi), \quad \eta = \infty, \tag{48}$$

which is the familiar "unfolded picture"[1] for Neumann and Dirichlet boundary conditions. The boundary conditions then forces nonzero mixed commutation relation [from Eqs. (A19) and (A20)] between right and left movers

$$[\varphi_R(\xi_1),\varphi_L(\xi_2)] = \begin{cases} -i\epsilon(\xi_{12}), & \eta=0, \\ i\epsilon(\xi_{12}), & \eta=\infty, \\ i\epsilon(\xi_{12}) - 4i\theta(\xi_{12})e^{-\eta\xi_{12}}, & 0<\eta<\infty, \end{cases} \tag{49}$$

while the left-left and right-right ones are the same as in the full line. Note that in the right-left commutators it appears $\xi_{12}=vt_{12}-\tilde{x}_{12}$, involving, as expected, the sum of distances from the boundary $\tilde{x}_{12}=x_1+x_2$.

Although right and left modes are no longer decoupled, we can still perform the bosonization program and solve the TL model on the half line. The anyonic exchange relations (1) are still valid defining ψ_α as in Eqs. (6) and (7) [but with normalization constants depending on the boundary conditions, see Eq. (A18)]. ψ_α fulfills the quantum equations of motion of the TL model restricted to the half line $x>0$, with σ, τ, and v given by the same expressions (18) and (19) found for the full line. In fact, all the local bulk relations of the TL model on the full line still hold on the half line. This will remain true in the more general case of a junction made of any number of wires.

The charge and current densities Eqs. (9) and (11) are still locally conserved [i.e., Eq. (10) holds for $x \neq 0$] and $\rho_\pm$ generate the $U(1)\otimes\tilde{U}(1)$ infinitesimal transformations (A16). After bosonization, the boundary condition (45) can be recasted in terms of physical currents

$$j_+(t,0)=0, \quad \eta=0,$$

$$j_-(t,0)=0, \quad \eta=\infty,$$

$$\partial_x j_-(t,0) - \eta j_-(t,0)=0, \quad 0<\eta<\infty. \tag{50}$$

Consequently, the main physical difference between half and full lines concerns the global charges Q and $\tilde{Q}$ associated to charge densities ρ_+ and ρ_-, respectively. The boundary spoils the simultaneous conservation of both charges, allowing just one linear combination to survive. For instance, at the critical point $\eta=0$, the boundary condition (45) is simply the Kirchhoff's rule associated to the $U(1)$ transformation (23), enforcing the charge-density current j_+ to vanish at the vertex while j_- does not,

$$j_+(t,0)=0, \quad j_-(t,0)\neq 0, \quad \text{for} \quad \eta=0. \tag{51}$$

In this case Q is time independent, while $\tilde{Q}$ depends on time due to a nontrivial charge flow through the boundary. The critical point $\eta=\infty$ has an opposite behavior, preserving the $\tilde{U}(1)$ transformation (24) and breaking (23). For generic finite $\eta>0$, it is easy to see that the $\tilde{U}(1)$ symmetry is always conserved while $U(1)$ is broken.[34] As already pointed out, this symmetry breaking from $U(1)\otimes\tilde{U}(1)$ to a subgroup $U(1)$ is a general unavoidable feature of junctions of any number of wires.

This boundary symmetry breaking is even more visible in the correlation functions. In addition to the usual right-right and left-left bosonic correlators, there are also mixed ones [Eqs. (A9), (A19), and (A20)]. As a consequence there are four nonvanishing two-point correlators for ψ_α, instead of just two as for the full line. For instance, considering the critical case $\eta=0$, when the $U(1)$ transformation (23) is preserved, we have

$$\begin{aligned}\langle\psi_1^*(t_1,x_1)\psi_1(t_2,x_2)\rangle &= \langle\psi_1(t_1,x_1)\psi_1^*(t_2,x_2)\rangle \\ &= [\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2} \\ &\quad\times[\mathcal{D}(vt_{12}-\tilde{x}_{12})\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\sigma\tau},\end{aligned} \tag{52}$$

$$\begin{aligned}\langle\psi_1^*(t_1,x_1)\psi_2(t_2,x_2)\rangle &= \langle\psi_2(t_1,x_1)\psi_1^*(t_2,x_2)\rangle \\ &= [\mathcal{D}(vt_{12}-x_{12})]^{\sigma\tau}[\mathcal{D}(vt_{12}+x_{12})]^{\sigma\tau} \\ &\quad\times[\mathcal{D}(vt_{12}-\tilde{x}_{12})]^{\sigma^2}[\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\tau^2},\end{aligned} \tag{53}$$

and

$$\begin{aligned}\langle\psi_2^*(t_1,x_1)\psi_2(t_2,x_2)\rangle &= \langle\psi_2(t_1,x_1)\psi_2^*(t_2,x_2)\rangle \\ &= (52) \quad \text{with} \quad \sigma\leftrightarrow\tau,\end{aligned} \tag{54}$$

$$\begin{aligned}\langle\psi_2^*(t_1,x_1)\psi_1(t_2,x_2)\rangle &= \langle\psi_1(t_1,x_1)\psi_2^*(t_2,x_2)\rangle \\ &= (53) \quad \text{with} \quad \sigma\leftrightarrow\tau,\end{aligned} \tag{55}$$

with $\tilde{x}_{12}=x_1+x_2$. The nontriviality of the correlators (53) and (55) reflects the breaking of the $\tilde{U}$ symmetry on the half line for $\eta=0$.

All the correlation functions just derived must be compared to the general scaling form coming from boundary conformal field theory[65] that in imaginary time $\tau_i=\mathrm{i}t_i$ predicts in general

$$\langle\Psi^*(z_1)\Psi(z_2)\rangle = \left(\frac{1}{z_{12}z_{\bar{1}\bar{2}}}\right)^{d_{\text{line}}} F(\xi), \tag{56}$$

with the four-point ratio

$$\xi = \frac{z_{1\bar{1}}z_{2\bar{2}}}{z_{1\bar{2}}z_{2\bar{1}}} \tag{57}$$

and $z_i=x_i+\mathrm{i}\tau_i$, $z_{\bar{i}}=\bar{z}_i$. $F(\xi)$ encodes all the boundary dependences and for small argument can be written as[65] $F(\xi\ll 1)\propto\xi^{d_b}$, where d_b is called boundary exponent. The real time correlations we wrote are clearly not of this form, but this is just because we wrote them in the regimes $x_1,x_2\gg 1$ and x_{12}, $\tilde{x}_{12}$ arbitrarily using definitions (6) and (7). If we want to get the correct scaling also for arbitrary $x_{1,2}$ we should modify the definitions as

$$\psi_1(t,x) \propto :e^{i\sqrt{\pi}\sigma\varphi_R(vt-x)}::e^{i\sqrt{\pi}\tau\varphi_L(vt+x)}:, \tag{58}$$

$$\psi_2(t,x) \propto :e^{i\sqrt{\pi}\tau\varphi_R(vt-x)}::e^{i\sqrt{\pi}\sigma\varphi_L(vt+x)}:, \tag{59}$$

at the price of introducing some more divergences that are easily renormalized. With this prescription, we obtain as a typical example

$$\begin{aligned}\langle\psi_1^*(t_1,x_1)\psi_1(t_2,x_2)\rangle &= \langle\psi_1(t_1,x_1)\psi_1^*(t_2,x_2)\rangle \\ &= [\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2} \\ &\quad\times\left[\frac{\mathcal{D}(vt_{12}-\tilde{x}_{12})\mathcal{D}(vt_{12}+\tilde{x}_{12})}{\mathcal{D}(2x_1)\mathcal{D}(2x_2)}\right]^{\sigma\tau},\end{aligned} \tag{60}$$

which agrees with the general conformal field theory scaling with $F(\xi)=\xi^{\sigma\tau}$ and so $d_b=\sigma\tau$. All the other correlation functions are easily modified accordingly. Because it will be easier to write, in the following, we will ignore the double normal product and still use definitions (6) and (7). The expressions taking into account the correct normalization at the boundary can be easily written down from the correlation we will derive.

We finally point out that for Dirichlet boundary conditions, i.e., $\eta=\infty$, the diagonal correlations are the same but with $d_b=-\sigma\tau$. Nondiagonal correlations can be found in Ref. 34.

C. Generic junction

The case of a junction with an arbitrary number $n>1$ of wires can be actually reduced to the study of a suitable family of n half lines. In fact, let $\mathcal{U}$ be the unitary matrix diagonalizing $\mathbb{U}$ which defines the boundary conditions (32). Since $\mathbb{U}$ is symmetric, we can choose $\mathcal{U}$ orthogonal, $\mathcal{U}^t = \mathcal{U}^{-1}$, and real, $\mathcal{U}^* = \mathcal{U}$. Let us parametrize the diagonal form

$$\mathbb{U}_d = \mathcal{U}\mathbb{U}\mathcal{U}^{-1} \tag{61}$$

as follows:

$$\mathbb{U}_d = \mathrm{diag}(e^{-2i\alpha_1}, e^{-2i\alpha_2}, \ldots, e^{-2i\alpha_n}). \tag{62}$$

Using definition (34) of $\mathbb{S}(k)$, one easily verifies that $\mathcal{U}$ diagonalizes $\mathbb{S}(k)$ for any k and that

$$\mathbb{S}_d(k) = \mathcal{U}\mathbb{S}(k)\mathcal{U}^{-1} = \mathrm{diag}\left(\frac{k-i\eta_1}{k+i\eta_1}, \frac{k-i\eta_2}{k+i\eta_2}, \ldots, \frac{k-i\eta_n}{k+i\eta_n}\right), \tag{63}$$

where

$$\eta_i = \lambda \tan(\alpha_i), \quad -\frac{\pi}{2} \le \alpha_i \le \frac{\pi}{2}. \tag{64}$$

Therefore $\mathbb{S}(k)$ is a meromorphic function in the complex k plane, whose poles are different from 0 and are all located on the imaginary axis. The condition (38) implies absence of bound states, i.e., of poles in the upper complex k plane, namely, $0 \le \alpha_i \le \pi/2$, hence $\eta_i \ge 0$.

Critical boundary conditions correspond to a matrix $\mathbb{U}$ such that the scattering matrix is insensitive to the momentum scale transformations $\lambda \to \varrho\lambda$ (or $k \to \varrho^{-1}k$) with $\varrho > 0$. To be scale invariant, the scattering matrix must have each η_i vanishing or infinite, so that S is actually momentum independent and with eigenvalues ± 1. By means of Eqs. (35) and (36), and the derivative[21]

$$k\frac{d\mathbb{S}(k)}{dk} = -\frac{1}{2}[\mathbb{S}(k) - \mathbb{S}^*(k)]\mathbb{S}(k), \tag{65}$$

we see that criticality is equivalent to the condition

$$\mathbb{S} = \mathbb{S}^*. \tag{66}$$

In Appendix B some examples of critical junctions with two, three, and four wires are given.

The matrix $\mathcal{U}$ allows us to define real scalar fields $\varphi^d = \mathcal{U}\varphi$ which are not localized on the single edges but have simple boundary conditions, formally the ones of disjoined half lines

$$(\partial_x \varphi^d)(t,0,i) - \eta_i \varphi^d(t,0,i) = 0, \quad i = 1, \ldots, n. \tag{67}$$

Comparing to the half line Eq. (49), it is straightforward to derive the commutation relations for the right and left movers on the wires as done in Refs. 20–22 and reported in Appendix A.

D. TL model at the junction

The TL model on the star graph Γ is defined by the sum of n Hamiltonians in Eq. (2) plus the boundary term that we implement through Eq. (32) at the bosonic level. The charges on each wire are defined via Eq. (3) and generate the $U(1) \otimes \tilde{U}(1)$ phase transformations (A16) and (A17) leaving the Hamiltonian invariant. The corresponding quantum equations of motion in the bulk are given by Eq. (5) for each wire independently.

In analogy with Eqs. (6) and (7), the solution of the equations of motion is given by the vertex operator

$$\psi_1(t,x,i) \propto :e^{i\sqrt{\pi}[\sigma\varphi_{i,R}(vt-x)+\tau\varphi_{i,L}(vt+x)]}:, \tag{68}$$

$$\psi_2(t,x,i) \propto :e^{i\sqrt{\pi}[\tau\varphi_{i,R}(vt-x)+\sigma\varphi_{i,L}(vt+x)]}:, \tag{69}$$

where the normalization constants are given in Appendix A and depend on the anyon Klein factors. All bulk relations (the value of σ, τ, and v, the form of the currents, etc.) of TL model on the line are still valid for half infinite wires jointed in a single vertex.

It is interesting to rewrite the boundary condition (32) in terms of physical quantities of the model: in particular at the critical point (66) where $\varphi_R(\xi) = S\varphi_L(\xi)$ (i.e., a generalized version of the unfolded picture of the half line), the boundary conditions get a very simple form

$$j_\pm(t,0,i) = \mp \sum_{j=1}^{n} \mathbb{S}_{ij} j_\pm(t,0,j), \tag{70}$$

which simply fixes the splitting of the currents at the junction. Comparing this expression to the Kirchhoff conditions (39) and (40), we see that at least one of two charges Q and $\tilde{Q}$, associated to ρ_+ and ρ_-, respectively, is dissipated by a nontrivial flow at the vertex. Since ρ_+ generates the electric charge for the ψ, Eq. (A16), we typically require the Kirchhoff's rule (39) to preserve electric charge, while $\tilde{Q}$ conservation is lost.

As for the half line, the nontrivial behavior of right-left correlators, due to the presence of vertex, allows more nonvanishing correlation functions with respect to the line case. Let us consider the two-point function for ψ in the Fock representation and let us focus for simplicity on the case of critical boundary conditions (66). Imposing the Kirchhoff's rule on the charge Q generated by $U(1)$, there are four nonvanishing two-point correlators

$$\begin{aligned}\langle \psi_1^*(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle &= \frac{z_{i_1}z_{i_2}}{2\pi}\Lambda^{-[(\sigma^2+\tau^2)\delta_{i_1i_2}+2\sigma\tau\mathbb{S}_{i_1i_2}]}\\ &\times[\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2\delta_{i_1i_2}}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2\delta_{i_1i_2}}\\ &\times[\mathcal{D}(vt_{12}-\tilde{x}_{12})\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\sigma\tau\mathbb{S}_{i_1i_2}},\end{aligned} \tag{71}$$

$$\begin{aligned}\langle \psi_1^*(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle &= \frac{z_{i_1}z_{i_2}}{2\pi}\Lambda^{-[(\sigma^2+\tau^2)\mathbb{S}_{i_1i_2}+2\sigma\tau\delta_{i_1i_2}]}\\ &\times[\mathcal{D}(vt_{12}-\tilde{x}_{12})]^{\sigma^2\mathbb{S}_{i_1i_2}}[\mathcal{D}(vt_{12}-x_{12})\mathcal{D}(vt_{12}+x_{12})]^{\sigma\tau\delta_{i_1i_2}}\\ &\times[\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\tau^2\mathbb{S}_{i_1i_2}},\end{aligned} \tag{72}$$

with all normalization factors defined in Appendix A. All other nonvanishing correlation functions have the same form as the ones on the half line Eqs. (52), (53), and (55) with only the proper wire index added.

For the charge densities one finds

$$\langle \rho_+(t_1,x_1,i_1)\rho_+(t_2,x_2,i_2)\rangle = \frac{-1}{(2\pi\zeta_+)^2}\{[\mathcal{D}^2(vt_{12}-x_{12})+\mathcal{D}^2(vt_{12}+x_{12})]\delta_{i_1i_2}+[\mathcal{D}^2(vt_{12}-\tilde{x}_{12})+\mathcal{D}^2(vt_{12}+\tilde{x}_{12})]\mathbb{S}_{i_1i_2}\}, \tag{73}$$

and for the currents

$$\langle j_+(t_1,x_1,i_1)j_+(t_2,x_2,i_2)\rangle = \frac{-1}{(2\pi\zeta_+)^2}\{[\mathcal{D}^2(vt_{12}-x_{12})+\mathcal{D}^2(vt_{12}+x_{12})]\delta_{i_1i_2}-[\mathcal{D}^2(vt_{12}-\tilde{x}_{12})+\mathcal{D}^2(vt_{12}+\tilde{x}_{12})]\mathbb{S}_{i_1i_2}\}. \tag{74}$$

The opposite signs in the $\delta_{i_1i_2}$ and $\mathbb{S}_{i_1i_2}$ contributions in Eq. (74) ensure the Kirchhoff's rule for Q. Analogous expressions hold for ρ_- and j_- up to replace in Eqs. (73) and (74) $(\tau+\sigma)\leftrightarrow(\tau-\sigma)$ and $\mathbb{S}\leftrightarrow-\mathbb{S}$.

If instead we impose the conservation of the charge $\tilde{Q}$ we have the nonvanishing two-point correlation functions

$$\langle \psi_1^*(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle = \frac{z_{i_1}z_{i_2}}{2\pi}\Lambda^{-[(\sigma^2+\tau^2)\delta_{i_1i_2}+2\sigma\tau\mathbb{S}_{i_1i_2}]}[\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2\delta_{i_1i_2}}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2\delta_{i_1i_2}}[\mathcal{D}(vt_{12}-\tilde{x}_{12})\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\sigma\tau\mathbb{S}_{i_1i_2}}, \tag{75}$$

$$\langle \psi_1(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle = \frac{z_{i_1}z_{i_2}}{2\pi}\Lambda^{[(\sigma^2+\tau^2)\mathbb{S}_{i_1i_2}+2\sigma\tau\delta_{i_1i_2}]}[\mathcal{D}(vt_{12}-x_{12})\mathcal{D}(vt_{12}+x_{12})]^{-\sigma\tau\delta_{i_1i_2}}[\mathcal{D}(vt_{12}-\tilde{x}_{12})]^{-\sigma^2\mathbb{S}_{i_1i_2}}[\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{-\tau^2\mathbb{S}_{i_1i_2}},$$

$$\langle \psi_1(t_1,x_1,i_1)\psi_1^*(t_2,x_2,i_2)\rangle = \langle \psi_1^*(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle, \tag{76}$$

$$\begin{aligned}\langle \psi_2(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle &= \langle \psi_2^*(t_1,x_1,i_1)\psi_1^*(t_2,x_2,i_2)\rangle\\ &= \langle \psi_1^*(t_1,x_1,i_1)\psi_2^*(t_2,x_2,i_2)\rangle\\ &= \langle \psi_1(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle,\end{aligned} \tag{77}$$

and

$$\begin{aligned}\langle \psi_2^*(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle &= \langle \psi_2(t_1,x_1,i_1)\psi_2^*(t_2,x_2,i_2)\rangle\\ &= \text{Eq. (75) with } \sigma\leftrightarrow\tau.\end{aligned} \tag{78}$$

The nonconservation of the electrical charge is explicitly shown by the presence of non-neutral correlator $\langle\psi\psi\rangle$. The correlations for conserved density ρ_- and current j_- are the same as Eqs. (73) and (74).

IV. RG FLOW ON THE JUNCTION

We completely characterized the fixed-point structure for a junction with an arbitrary number of wires n. Let us recall the main features explained in Sec. III and in Appendix B. At the critical point, the scattering matrix can only have eigenvalues ± 1. For generic n, the fixed points are classified in terms of the integer number p with $0\leq p\leq n$, which is the number of eigenvalues equal to -1. At the fixed point, the boundary couplings η_i (with $1\leq i\leq n$) are zero if the corresponding eigenvalue is +1 and infinity if the eigenvalue is -1. $p=0$ corresponds to Neumann boundary conditions on all wires, while $p=n$ to Dirichlet. Other values of p correspond to intermediate boundary conditions that are $n-p$ Neumann and p Dirichlet fields in the basis φ_i^d diagonalizing the S matrix. In Fig. 2 we report as a typical example the RG flow diagram for three wires in the η_i space. The final point of any axis is $\eta_i=\infty$. Let us discuss now the structure of the fixed points, postponing the study of the stability to the following. There are $2^3=8$ fixed-point families, one Neumann, three points with $p=1$, three with $p=2$, and one Dirichlet [in the general case, there are 2^n families of which $\binom{n}{p}$ for any

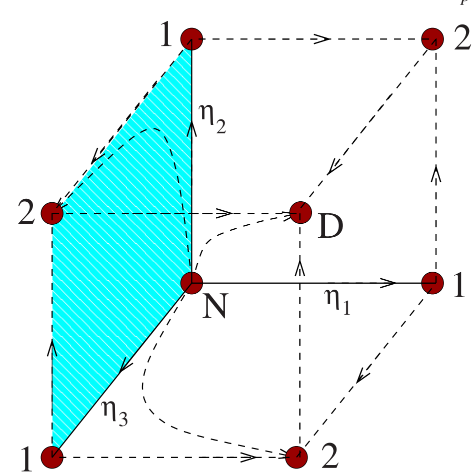

FIG. 2. (Color online) RG flow diagram for a junction of three wires in the $\eta_{1,2,3}$ space. The fixed points are: D is Dirichlet and corresponds to all $\eta_i=\infty$; N is Neumann with $\eta_i=0$; 1 are three fixed-point families (depending on two parameters) with two η vanishing and one infinite; 2 are three fixed-point two-parameter families with one η zero and two infinite. The cyan-shaded area is the allowed region when the Kirchhoff's rule for the electric charge is valid. It includes Neumann, two $p=1$ families (with only one parameter left free), and one $p=2$ fixed point (with no free parameter left). The arrow in the flow corresponds to the attractive case with $g>1$ which gives Dirichlet as the most stable fixed point (without Kirchhoff) or the mixed $p=2$ (with Kirchhoff). In the opposite repulsive case $g<1$, all the arrows are reversed and the most stable fixed point is Neumann.

p]. Every critical point belongs to a continuous family with $p(n-p)$ real parameters that are not shown in Fig. 2. Summarizing any critical point is identified by p, by the specific eigenvalues that are -1 (i.e., by the axis in the figure) and by the $p(n-p)$ real parameters. The parameters specifying the fixed point in the families are the angles α_i reported for some examples in Appendix B. For a given situation, the fixed-point value of α_i is given by their initial values. This means that α_i are marginal couplings and their values cannot be fixed only by requiring scale invariance.

The role played by the conservation rules in this flow diagram is fundamental. To consider the most physical case, let us discuss when the electrical charge is conserved, i.e., the Kirchhoff rule $\Sigma_i j_+(t,0,i)=0$ is satisfied. The first effect is to fix to zero one (arbitrary) η_i, constraining the system on the shadow area in Fig. 2 so that Dirichlet boundary conditions are ruled out for the problem. Also the number of real parameters characterizing the p fixed points is largely reduced. For three wires, the point with $p=1$ becomes a one-parameter family, while the point with $p=2$ becomes an isolated fixed point. Details for the general case are in Appendix B. Needless to say that imposing the conservation of the $\tilde{U}(1)$ charge results in fixing one of the η_i to ∞ and similarly reduced the number of real parameters available for each fixed point.

We briefly discuss our terminology for the fixed points in order to make the comparison to other papers as simple as possible. The fixed points with $p=2$ ("2" in Fig. 2) is the mixed fixed point found by Nayak *et al.*[1] and called D (or D_P) in Ref. 9 because of the $n-1$ Dirichlet boundary conditions (there $n=3$) on the neutral modes (but this point is obviously different from our D). The family with $p=1$ in Fig. 2 depends on a continuous real parameter α, as shown in Eq. (B4), and it has been first found in Ref. 21. Note that it is not symmetrical under wire permutations. There are three special values of α: for $\alpha=-1,0,\infty$ the S matrix breaks into 1×1 and 2×2 blocks. The 1×1 block is a wire decoupled from the other two that form a purely transmitting $n=2$ junction (the same can be verified for higher n; changing the α's we can decouple any wire). For these special values of α, the fixed points were also found by Chamon *et al.*[9] that called them asymmetrical D_A. Other values of α interpolate continuously between these three. Finally it is worth commenting that the Dirichlet fixed point (D in Fig. 2) physically corresponds to n wires with an end inserted into a large superconductor. In fact, the S matrix $S=-\mathbb{I}$ gives conductance $\mathbb{G}=2\mathbb{I}$ corresponding to Andreev reflection in all wires (i.e., sending a particle one gets a hole out). This is a different problem from a junction of wires (even superconducting) because the large superconductor breaks the $U(1)$ charge conservation.[9,25]

Now we know the fixed-point structure, but what is the relative stability? Which fixed point describe the universal low-energy behavior? There are several equivalent ways to tackle this question. The more natural one, as done elsewhere,[1,9,23] relies on calculating the scaling dimension of the perturbing operator at a given fixed point. Since our problem can be thought as n independent half lines with $n-p$ Neumann boundary conditions and p Dirichlet ones, the problem is just equivalent to understand the stability of Neumann or Dirichlet against a Robin term as in the Hamiltonian (46). This is a standard problem. In the bosonic theory, the flow can be followed exactly from Eq. (65) of the off-critical S matrix. The Neumann fixed point is always unstable, while Dirichlet is stable (or mixed if Kirchhoff is imposed on the electrical charge). However, as well known, considering the fermionic theory changes this scenario because of the Klein factors. In boundary conformal field theory, the stability conditions are just read from the boundary dimensions d_b appearing in the two-point correlation functions reported above. At the Neumann boundary condition (BC) we have the dimension $\sigma\tau=(\zeta_+^2-\zeta_-^2)/4$ that is greater than zero for $g_+>g_-$, i.e., for repulsive anyonic interaction, giving a stable Neumann. Oppositely at the Dirichlet BC the boundary dimension is $-\sigma\tau$ that it is stable in the complementary attractive case. Since there are no other fixed points in the RG diagrams, this analysis fixes all the RG flow. Note that for free anyons (and in particular fermions) η is marginal in this approach. In any given anyonic or fermionic model the actual stable fixed point will be determined by the higher order terms in η neglected in our approach.

These results can be confirmed on the basis of the following argument based on the so-called g theorem.[67,68] For a one-dimensional critical system with a boundary, it is known that the boundary contribution to the entropy $\ln g$ (g is the so-called "universal noninteger ground-state degeneracy"[67]) decreases along the renormalization-group flow. We can easily calculate the value of the effective potential $V_{\text{eff}}=g_+\rho_+ + g_-\rho_-$ for the off critical model for any η. Subtracting the divergent contribution of the bulk to make this expectation value finite, we get on each wire

$$\varepsilon(x,i)=\langle V_{\text{eff}}(t,x,i)\rangle=\Omega\int_{-\infty}^{+\infty}\frac{dk}{2\pi}|k|e^{2ikx}\mathbb{S}_{ii}(k), \tag{79}$$

where

$$\Omega=\left(\frac{g_-\zeta_-^2-g_+\zeta_+^2}{2\pi\kappa^2}\right) \tag{80}$$

fully encodes the bulk interactions effect. In particular, when $g_+=g_-$ it vanishes and changes sign, giving the correct stability scenario.

In fact, we can rewrite Eq. (79) in terms of the potential $\varepsilon_{\eta_j}(x)$ for disjointed half line with the boundary condition (67)

$$\varepsilon(x,i)=\sum_j^n|\mathcal{U}_{ji}|^2\varepsilon_{\eta_j}(x), \tag{81}$$

with

$$\varepsilon_\eta(x)=-\frac{\Omega}{4x^2}[1-4(x\eta)-8(x\eta)^2e^{2x\eta}\text{Ei}(-2\times\eta)]. \tag{82}$$

The function

$$s(x) = -4x^2\sum_{i=1}^{n} \varepsilon(x,i) = -4x^2\sum_{j=1}^{n} \varepsilon_{\eta_j}(x) \tag{83}$$

collects the contribution of all the wires. It is a monotonous function with fixed points at $\eta=0,\infty$ in agreement with the g theorem. The stability of the fixed points and the direction of the flow are just given by the sign of Ω and agree with the previous analysis.

V. CONCLUSIONS

In this paper we presented a systematic study of the critical properties of n anyonic Luttinger wires jointed in a single vertex. Imposing the boundary conditions (32) at the junction directly on bosonized fields allowed us to describe completely the RG flow diagram for any n. As a typical example the RG flow for $n=3$ is depicted in Fig. 2 where the main features of the various fixed points are discussed in the text.

At this point it is worth comparing our findings with the literature. For two wires, our results are a simple anyonic generalization of the well-known ones by Kane and Fisher[29] for fermions that are reproduced for $\kappa=1$. For $n=3$, as we said in Sec. I, the literature is enormous. The boundary conditions we used are equivalent to those of the "auxiliary model" of Nayak *et al.*[1] for $g\neq 1$ [in fact, expanding the exponential defining the auxiliary model[1] and keeping only up to the quadratic terms, neglecting irrelevant higher orders, we arrive to the Hamiltonian (46) where the symmetry of the boundary terms is just the Kirchhoff's rule]. We predict two possible stable fixed points: Neumann and mixed. Neumann is well known, it has zero conductance, and in this setting it is stable for all repulsive interaction, i.e., $g<1$. The mixed fixed point has been found for the first time by Nayak *et al.*[1] and it is specific of the junctions. It has enhanced conductance $G/G_{\text{line}}=4/3$ and we found it is stable for all attractive interactions $g>1$ as in Ref. 1. Everything agrees with the auxiliary model, but not with the "standard model" defined by the boundary condition (4) that is known to be different.[1] In fact in the standard model, the Neumann fixed point is stable only for $g<1/3$ while the mixed one only for $g>9$. In the other regimes with $1/3<g<9$, new fixed points appear that *cannot* be present in our approach.[1,9] Our setting however presents a great advantage: it is simpler for generic n and more efficient in describing the off-critical properties of the system. In fact we provide the critical behavior for all n. We found for $g<1$ a Neumann stable fixed point (with zero conductance) and for $g>1$ a mixed fixed point with conductance $G/G_{\text{line}}=2(n-1)/n$. We also find other fixed points (described in Appendix B) that however have at least one direction of instability in the η_i space and so they are multicritical points in the sense that some other constraints must be imposed to reach them. Clearly we expect that the standard model for $n\geq 3$ will have some fixed points not found here, as for the case $n=3$. A part from the *per se* interest of the model, the fixed points we found are relevant for the standard model as well. In fact, it is easy to generalize to any n the strong and weak boundary coupling (i.e., our η) calculations of Refs. 1 and 9 to show that for small enough g the relevant fixed point is Neumann and for large enough is the mixed one. However, which fixed point governs the dynamics when none of these two is stable is not accessible to our approach. For $n=4$, two fixed points derived in Appendix B have been recently found to describe the scattering matrix for a proposed experiment to detect the helical nature of the edge states in quantum Hall systems.[26]

We mention that we also characterized the junction in the absence of the Kirchhoff's rule for the electric charge. It is of particular relevance considering the case when relaxing the conservation of the electrical charge and imposing the conservation of the dual one $\widetilde{U}(1)$. In this case the more stable fixed point is always Dirichlet with uncommon points such as the mixed one representing multicritical points.

There are two generalizations of the model considered here that should be easily accessible to a similar analysis. First of all one can consider fermions with spin (and even multispecies anyons) as done elsewhere with fermionic boundary conditions.[23] In this way one can understand which fixed points are present also with bosonic boundary conditions. The other generalization is relaxing the symmetry for time reversal to allow a nonvanishing flux at the junction.[9]

We close this paper on a more speculative level. In recent times there has been an increasing interest in quantifying the entanglement in extended quantum systems (see, e.g., Ref. 69 as review). Among the various measures, the so-called entanglement entropy has by far been the most studied. By partitioning an extended quantum system into two blocks, the entanglement entropy is defined as the von Neumann entropy of the reduced density matrix ρ_A of one of the two blocks. This procedure requires an arbitrary division of the system in two parts. In the junction problem studied here the system is automatically divided in parts and it would be very interesting to understand the amount of entanglement between the various wires. The analysis of some models on the line with one defect[70] (i.e., $n=2$ in the language of this paper) showed that the entanglement entropy is not always only dependent on the central charge of the bulk theory (as maybe naively expected). The natural question is whether the conformal field theory formalism that has been successfully applied to the bulk and boundary cases[71] can be generalized to the junction. Furthermore, if we would be able to solve the nonequilibrium problem with changing the boundary condition (e.g., suddenly adding or removing the junctions, as done for $n=2$ in Ref. 72), one can think of using the junction as an *entanglement meter* following the recent proposal based on quantum noise measurement.[73]

ACKNOWLEDGMENTS

We thank Claudio Chamon for fruitful comments on a first version of the paper and for useful discussions. P.C. benefited by a travel grant from ESF (INSTANS activity).

APPENDIX A: BONSONIZATION AND QUANTIZATION OF THE TL MODEL

1. Line

The basic tool for quantizing the system, described by Eq. (5), is the algebra $\mathcal{A}$ generated by the bosonic annihilation $a(k)$ and creation $a^*(k)$ operators satisfying

$$[a(k),a(p)]=[a^*(k),a^*(p)]=0, \tag{A1}$$

$$[a(k),a^*(p)]=4\pi|k^{-1}|_\Lambda\delta(k-p), \tag{A2}$$

where the normalization can be fixed such that

$$|k^{-1}|_\Lambda=\frac{d}{dk}\left[\theta(k)\ln\frac{ke^{\gamma_E}}{\Lambda}\right]. \tag{A3}$$

The derivative here is understood in the sense of distributions, γ_E is Euler's constant, and $\Lambda>0$ is a free parameter with dimension of mass having a well-known infrared origin. It is useful to introduce

$$u(\Lambda\xi)\equiv\int_0^\infty\frac{dk}{\pi}|k^{-1}|_\Lambda e^{-ik\xi}=-\frac{1}{\pi}\ln(\Lambda|\xi|)-\frac{i}{2}\varepsilon(\xi)=-\frac{1}{\pi}\ln(i\Lambda\xi+\epsilon),\quad \epsilon>0. \tag{A4}$$

The left and right chiral fields are defined by

$$\varphi_R(\xi)=\int_0^\infty\frac{dk}{2\pi}[a^*(k)e^{ik\xi}+a(k)e^{-ik\xi}], \tag{A5}$$

$$\varphi_L(\xi)=\int_0^\infty\frac{dk}{2\pi}[a^*(-k)e^{ik\xi}+a(-k)e^{-ik\xi}], \tag{A6}$$

obey the commutation relations

$$[\varphi_R(\xi_1),\varphi_R(\xi_2)]=[\varphi_L(\xi_1),\varphi_L(\xi_2)]=-\mathrm{i}\varepsilon(\xi_{12}), \tag{A7}$$

$$[\varphi_R(\xi_1),\varphi_L(\xi_2)]=0, \tag{A8}$$

and have the correlations

$$\langle\varphi_R(\xi_1)\varphi_R(\xi_2)\rangle=\langle\varphi_L(\xi_1)\varphi_L(\xi_2)\rangle=u(\Lambda\xi_{12}), \tag{A9}$$

with $\xi_{12}=\xi_1-\xi_2$ and obviously $\langle\varphi_R(\xi_1)\varphi_L(\xi_2)\rangle=0$.

Defining the chiral charges by

$$Q_Z=\frac{1}{4}\int_{-\infty}^{\infty}d\xi(\partial\varphi_Z)(\xi),\quad Z=R,L, \tag{A10}$$

one gets

$$[Q_R,\varphi_R(\xi)]=[Q_L,\varphi_L(\xi)]=-i/2,$$

$$[Q_R,\varphi_L(\xi)]=[Q_L,\varphi_R(\xi)]=[Q_R,Q_L]=0. \tag{A11}$$

It is worth mentioning that all previous commutation relations are invariant under the *duality* transformation

$$\varphi_R(\xi)\mapsto\varphi_R(\xi),\quad \varphi_L(\xi)\mapsto-\varphi_L(\xi), \tag{A12}$$

which define the T duality in string theory.

At this point we are ready to introduce a family of vertex operators parametrized by two real variables σ and τ defined by

$$A(t,x)=z\mathrm{e}^{\mathrm{i}\sqrt{\pi}(\tau Q_R-\sigma Q_L)}:\mathrm{e}^{\mathrm{i}\sqrt{\pi}[\sigma\varphi_R(vt-x)+\tau\varphi_L(vt+x)]}:, \tag{A13}$$

with

$$z=(2\pi)^{-1/2}\Lambda^{(\sigma^2+\tau^2)/2}, \tag{A14}$$

where $:\cdots:$ denotes the normal product in $\mathcal{A}$ and v is some velocity to be determined by consistency. From Eqs. (6) and (7) the fields ψ_1 and ψ_2 are vertex operators with interchanged σ and τ, with a normalization constant given by Eq. (A14). The factor $e^{i\sqrt{\pi}(\tau Q_R-\sigma Q_L)}$ is included in the definition (A14) to ensure canonical anionic commutation relation between $\psi_{1,2}$ without introducing Klein factors that will be important only for the fields on different wires.

The following identity is useful in determining the exchange properties of the vertex operators and so all correlation functions

$$A^*(t,x_1)A(t,x_2)=|x_{12}|^{-(\sigma^2+\tau^2)}e^{-i\pi/2(\tau^2-\sigma^2)\varepsilon(x_{12})}:e^{i\sqrt{\pi}[\sigma\varphi_R(vt-x_2)-\sigma\varphi_R(vt-x_1)+\tau\varphi_L(vt+x_2)-\tau\varphi_L(vt+x_1)]}:, \tag{A15}$$

where $x_{12}\equiv x_1-x_2$.

The normalization of the charge densities $\rho_\pm$ is fixed by requiring that they generate the transformations (24) and (23) in infinitesimal form, namely,

$$[\rho_+(t,x_1),\psi_\alpha(t,x_2)]=-\delta(x_{12})\psi_\alpha(t,x_2), \tag{A16}$$

$$[\rho_-(t,x_1),\psi_\alpha(t,x_2)]=-(-1)^\alpha\delta(x_{12})\psi_\alpha(t,x_2). \tag{A17}$$

2. Half line

In the main text, we stressed that on the half line right and left modes couple and have nontrivial commutation relations given by Eq. (49). This gives rise to few changes to the relations valid on the full line. The vertex operator is always defined by Eq. (A13), but the normalization constant is affected by the boundary[34]

$$z=\begin{cases}(2\pi)^{-1/2}\Lambda^{(\sigma+\tau)^2/2}, & \eta=0;\\ (2\pi)^{-1/2}\Lambda^{(\sigma-\tau)^2/2}, & 0<\eta\leq\infty.\end{cases} \tag{A18}$$

The right-left coupling also affects the correlation functions of the field φ. In fact, while the right-right and left-left correlators are still given by Eq. (A9), the mixed ones are

$$\langle\varphi_R(\xi_1)\varphi_L(\xi_2)\rangle=\begin{cases}u(\Lambda\xi_{12}) & \eta=0,\\ -u(\Lambda\xi_{12}) & \eta=\infty,\\ -u(\Lambda\xi_{12})-v_-(\Lambda\xi_{12}) & 0<\eta<\infty,\end{cases} \tag{A19}$$

$$\langle \varphi_L(\xi_1)\varphi_R(\xi_2)\rangle = \begin{cases} u(\Lambda\xi_{12}) & \eta=0, \\ -u(\Lambda\xi_{12}) & \eta=\infty, \\ -u(\Lambda\xi_{12})-v_+(\Lambda\xi_{12}) & 0<\eta<\infty, \end{cases} \tag{A20}$$

where the "boundary propagator" is

$$v_\pm(\xi) = \frac{2}{\pi}e^{-\xi}\mathrm{Ei}(\xi \pm i\epsilon), \tag{A21}$$

and $\mathrm{Ei}(x)=\int_x^\infty dz e^{-z}/z$ is the exponential integral function that at small x has the right logarithm expansion. Note that in the above formulas for mixed correlators $\xi_1=vt_1-x_1$ and $\xi_2=vt_2+x_2$ or vice versa, thus $\xi_{12}=vt_{12}\mp\tilde{x}_{12}$, with the sign depending on the correlator if it is right-left or left-right, respectively.

3. Junction

For the theory on the star graph, all the relevant commutation relations and correlators of the fields follow from those on the half line after performing the linear transformation $\mathcal{U}$ in Eq. (61). In fact all the fields φ^d are just delocalized fields satisfying the proper boundary conditions reported above with different η_i for each mode. Thus, comparing to the half line Eq. (49), it is straightforward to derive the commutation relations for the right and left movers on the wires

$$[\varphi_{i_1,R}(\xi_1),\varphi_{i_2,R}(\xi_2)] = [\varphi_{i_1,L}(\xi_1),\varphi_{i_2,L}(\xi_2)] = -i\epsilon(\xi_{12})\delta_{i_1i_2}, \tag{A22}$$

$$[\varphi_{i_1,R}(\xi_1),\varphi_{i_2,L}(\xi_2)] = \mathcal{U}^{-1}_{i_1k_1}\mathcal{U}_{l_2i_2}[\varphi^d_{k_1,R}(\xi_1),\varphi^d_{l_2,L}(\xi_2)], \tag{A23}$$

where $\varphi^d_{R,L}(\xi)=\mathcal{U}\varphi_{R,L}(\xi)$ and

$$[\varphi^d_{i_1,R}(\xi_1),\varphi^d_{i_2,L}(\xi_2)] = \begin{cases} -i\epsilon(\xi_{12})\delta_{i_1i_2}, & \eta_{i_1}=0; \\ i\epsilon(\xi_{12})\delta_{i_1i_2}, & \eta_{i_1}=\infty; \\ [i\epsilon(\xi_{12})-4i\theta(\xi_{12})e^{-\eta_{i_1}\xi_{12}}]\delta_{i_1i_2}, & 0<\eta_{i_1}<\infty. \end{cases} \tag{A24}$$

The mixed commutator (A23) simplifies greatly for critical boundary conditions

$$[\varphi_{i_1,R}(\xi_1),\varphi_{i_2,L}(\xi_2,i_2)] = -i\epsilon(\xi_{12})\mathbb{S}_{i_1i_2}. \tag{A25}$$

Note that at spacelike distances where $vt_{12}-\tilde{x}_{12}<0$, the commutators (A22) and (A23) behave as if the scattering matrix were replaced by the critical one obtained in the infrared limit $\Lambda\to\infty$ or equivalently $k\to 0$,

$$[\varphi_{i_1,R}(\xi_1),\varphi_{i_2,L}(\xi_2)]|_{(v^2t_{12}^2-x_{12}^2<0)} = -i\mathbb{S}_{i_1i_2}(0). \tag{A26}$$

This simply means that $\varphi_{R,L}$ has the same properties of locality than its infrared limit.

The last complication on the star graph arises in the definition of the anyonic fields $\psi_{1,2}$. To have the correct commutation relation they must be defined according to

$$\psi_1(t,x,i) = z_i\eta_i e^{i\sqrt{\pi}(\tau Q_{i,R}-\sigma Q_{i,L})}:e^{i\sqrt{\pi}[\sigma\varphi_{i,R}(vt-x)+\tau\varphi_{i,L}(vt+x)]}:,$$

$$\psi_2(t,x,i) = z_i\eta_i e^{i\sqrt{\pi}(\sigma Q_{i,R}-\tau Q_{i,L})}:e^{i\sqrt{\pi}[\tau\varphi_{i,R}(vt-x)+\sigma\varphi_{i,L}(vt+x)]}:, \tag{A27}$$

where z_i are fixed to

$$z_i = (2\pi)^{-1/2}\Lambda^{[(\sigma^2+\tau^2)+2\sigma\tau\mathbb{S}_{ii}(0)]/2} \tag{A28}$$

and η_i are the anyonic Klein factors needed to ensure the correct commutation of anyon fields on different edges

$$\psi(t,x_i,i)\psi(t,x_j,j) = e^{-i\pi\kappa\epsilon_{ij}}\psi(t,x_j,j)\psi(t,x_i,i), \tag{A29}$$

where $\epsilon_{ij}=\epsilon(i-j)$. It is straightforward to build them, for example, in terms of the auxiliary Majorana algebra $[c_i,c_j]=i\kappa\epsilon_{ij}$ and $c_i^*=c_i$ resulting in $\eta_i=:e^{\pi i c_i}:$. These factors are of fundamental importance when considering as junction condition Eq. (4) because it is written in terms of anyonic degrees of freedom. Oppositely, because the junction condition we use is written in terms of currents that only get (re)normalized by the statistics, they are inessential. For this reason we do not discuss them further, remanding the interested readers to the complete treatment presented in Ref. 74 and in the appendix E of Ref. 9.

APPENDIX B: CRITICAL POINTS

By scale invariance any critical point is associated with a k-independent S matrix satisfying unitarity (35), Hermitian analyticity (36), and time-reversal invariance (37), i.e.,

$$\mathbb{S}^* = \mathbb{S}^{-1}, \quad \mathbb{S}^* = \mathbb{S}, \quad \mathbb{S}^t = \mathbb{S}. \tag{B1}$$

The classification of these S matrices is now a simple matter. Indeed, one can easily deduce from Eq. (B1) that the eigenvalues of $\mathbb{S}$ are ± 1. Let us denote by p the number of eigenvalues -1. The values $p=0$ and $p=n$ correspond to the familiar Neumann ($\mathbb{S}_N=\mathbb{I}$) and Dirichlet ($\mathbb{S}_D=-\mathbb{I}$) boundary conditions, respectively. A richer structure appears for $0<p<n$. In that case the S matrices satisfying Eq. (B1) depend on $p(n-p)\geq 1$ parameters, giving raise to whole families of critical points.[20,21] Let us describe them explicitly for $n=2$, 3, 4.

The only possibility for $n=2$ is $p=1$, leading to the one-parameter family[39,40]

$$\mathbb{S} = \frac{1}{1+\alpha^2}\begin{pmatrix} \alpha^2-1 & -2\alpha \\ -2\alpha & 1-\alpha^2 \end{pmatrix}, \tag{B2}$$

with α a real number. For $\alpha=-1$ one has full transmission and no reflection, which corresponds to the theory on the whole line. This is an example of exceptional boundary conditions already mentioned.[66] It is only the only S matrix in the family satisfying Kirchhoff's rule for the electric charge. Oppositely, $\alpha=1$ is the only matrix satisfying Kirchhoff's rule for the $\tilde{U}(1)$ charge, as predicted by duality.

In the case $n=3$ one has two possibilities: $p=2$ and $p=1$. In both cases one has a family with two real parameters $\alpha_{1,2}$,

$$\mathbb{S}_2(\alpha_1,\alpha_2) = \frac{1}{1+\alpha_1^2+\alpha_2^2} \times \begin{pmatrix} \alpha_1^2-\alpha_2^2-1 & 2\alpha_1\alpha_2 & 2\alpha_1 \\ 2\alpha_1\alpha_2 & -\alpha_1^2+\alpha_2^2-1 & 2\alpha_2 \\ 2\alpha_1 & 2\alpha_2 & 1-\alpha_1^2-\alpha_2^2 \end{pmatrix} \tag{B3}$$

and

$$\mathbb{S}_1(\alpha_1,\alpha_2) = -\mathbb{S}_2(\alpha_1,\alpha_2). \tag{B4}$$

For generic values of the parameters these S matrices violate both $U(1)$ and $\tilde{U}(1)$. Preserving $U(1)$, one must impose Eq. (39) on Eq. (B3). This implies $\alpha_1=\alpha_2=1$, leading to the isolated critical point

$$\mathbb{S}_2 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{B5}$$

which is invariant under edge permutations. From Eq. (B4) one obtains instead $\alpha_2=-(1+\alpha_1)$. Therefore, setting $\alpha\equiv\alpha_2$, one has in this case the one-parameter family of critical points

$$\mathbb{S}_1 = \frac{1}{1+\alpha+\alpha^2}\begin{pmatrix} -\alpha & \alpha(\alpha+1) & 1+\alpha \\ \alpha(\alpha+1) & \alpha+1 & -\alpha \\ \alpha+1 & -\alpha & \alpha(\alpha+1) \end{pmatrix}, \tag{B6}$$

which is *not* invariant under edge permutations for generic α. Summarizing, the critical points which respect $U(1)$ are $\mathbb{S}_0=\mathbb{I}_3$ and Eqs. (B5) and (B6). The matrix (B5) has been discovered by means of RG techniques by Nayak *et al.*[1] The family (B6) appeared for the first time in Ref. 21.

If one wants to preserve $\tilde{U}(1)$, one must require Eq. (40). One is left therefore with $\mathbb{S}_3=-\mathbb{I}_3$,

$$\mathbb{S}_2 = -\frac{1}{1+\alpha+\alpha^2}\begin{pmatrix} -\alpha & \alpha(\alpha+1) & 1+\alpha \\ \alpha(\alpha+1) & \alpha+1 & -\alpha \\ \alpha+1 & -\alpha & \alpha(\alpha+1) \end{pmatrix}, \tag{B7}$$

and

$$\mathbb{S}_1 = -\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{B8}$$

as predicted by duality.

For $n=4$ the general matrices satisfying all the constraints (B1) are too large to be reported here. Thus we only give the critical points for $n=4$ satisfying the Kirchhoff's rule Eq. (39) for the electrical current [the analogous ones with the Kirchhoff's rule Eq. (39) are just $-\mathbb{S}$ because of duality]. Besides $\mathbb{S}_0=\mathbb{I}_4$ corresponding to $p=0$, one has:

(i) for $p=1$ the $\mathbb{S}$ matrix depends on two real parameters $\alpha_{1,2}$ and results to be

$$\mathbb{S}_{11} = \frac{1}{\Delta_1}(\alpha_1+\alpha_1^2+\alpha_2+\alpha_1\alpha_2+\alpha_2^2),$$

$$\mathbb{S}_{22} = \frac{1}{\Delta_1}(1+\alpha_1+\alpha_1^2+\alpha_2+\alpha_1\alpha_2),$$

$$\mathbb{S}_{33} = \frac{1}{\Delta_1}(1+\alpha_1+\alpha_2+\alpha_1\alpha_2+\alpha_2^2),$$

$$\mathbb{S}_{44} = -\frac{1}{\Delta_1}(\alpha_1+\alpha_2+\alpha_1\alpha_2),$$

$$\mathbb{S}_{12} = -\frac{1}{\Delta_1}\alpha_2, \quad \mathbb{S}_{13} = -\frac{1}{\Delta_1}\alpha_1,$$

$$\mathbb{S}_{14} = \frac{1}{\Delta_1}(1+\alpha_1+\alpha_2),$$

$$\mathbb{S}_{23} = -\frac{1}{\Delta_1}\alpha_1\alpha_2, \quad \mathbb{S}_{24} = \frac{1}{\Delta_1}\alpha_2(1+\alpha_1+\alpha_2),$$

$$\mathbb{S}_{34} = \frac{1}{\Delta_1}\alpha_1(1+\alpha_1+\alpha_2),$$

with $\Delta_1=1+\alpha_1+\alpha_1^2+\alpha_2+\alpha_1\alpha_2+\alpha_2^2$. The remaining entries are recovered by symmetry. Note that this matrix is not invariant under edge permutations.

(ii) For $p=2$ the S matrix still depends on two real parameters

$$\mathbb{S}_{11} = \frac{1}{\Delta_2}[3\alpha_1^2+2\alpha_1(1-\alpha_2)-(1+\alpha_2)^2],$$

$$\mathbb{S}_{22} = \frac{1}{\Delta_2}[-1-\alpha_1^2+2\alpha_2+3\alpha_2^2-2\alpha_1(1+\alpha_2)],$$

$$\mathbb{S}_{33} = \frac{1}{\Delta_2}[3-\alpha_1^2+2\alpha_2-\alpha_2^2+2\alpha_1(1+\alpha_2)],$$

$$\mathbb{S}_{44} = -\frac{1}{\Delta_2}[\alpha_1^2+2\alpha_1(1-\alpha_2)+(1+\alpha_2)^2],$$

$$\mathbb{S}_{12} = \frac{2}{\Delta_2}(1 + \alpha_1 + \alpha_2 + 2\alpha_1\alpha_2),$$

$$\mathbb{S}_{13} = \frac{2}{\Delta_2}[\alpha_2(1 + \alpha_2) - \alpha_1(2 + \alpha_2)],$$

$$\mathbb{S}_{14} = \frac{2}{\Delta_2}(1 + \alpha_1 - \alpha_1\alpha_2 + \alpha_2^2),$$

$$\mathbb{S}_{23} = \frac{2}{\Delta_2}(\alpha_1 + \alpha_1^2 - 2\alpha_2 - \alpha_1\alpha_2),$$

$$\mathbb{S}_{24} = \frac{2}{\Delta_2}(1 + \alpha_1^2 + \alpha_2 - \alpha_1\alpha_2),$$

$$\mathbb{S}_{34} = \frac{2}{\Delta_2}(\alpha_1 + \alpha_1^2 + \alpha_2 + \alpha_2^2),$$

where $\Delta_2=3+3\alpha_1^2+2\alpha_1(1-\alpha_2)+2\alpha_2+3\alpha_2^2$. Also this matrix is not invariant under edge permutations.

(iii) For $p=3$ we have only an isolated S matrix

$$\mathbb{S} = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -2 \end{pmatrix}, \tag{B9}$$

which is invariant under edge permutation. This is the analogous for four wires of the Nayak *et al.*[1] result.

Recently, the $p=1$ matrix, with $\alpha_1=1$ and $\alpha_2=-1$, and the $p=3$ matrix have been found to describe the scattering matrix for a proposed experiment to detect the helical nature of the edge states in quantum Hall systems.

We conclude this appendix with the matrix with $p=n-1$ for general n satisfying the electric Kirchhoff rule

$$\mathbb{S} = \frac{1}{n}\begin{pmatrix} (2-n) & 2 & 2 & \cdots & 2 \\ 2 & (2-n) & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 2 & 2 & \cdots & (2-n) \end{pmatrix}, \tag{B10}$$

since it is the most stable in the RG phase diagram as shown in the text. This is the only matrix which is invariant under wire permutations (i.e., that has all diagonal elements equal and nondiagonal as well), satisfying the Kirchhoff's rule and with all nonvanishing entries.

[1] C. Nayak, M. P. A. Fisher, A. W. W. Ludwig, and H. H. Lin, Phys. Rev. B **59**, 15694 (1999).
[2] I. Safi, P. Devillard, and T. Martin, Phys. Rev. Lett. **86**, 4628 (2001).
[3] J. E. Moore and X.-G. Wen, Phys. Rev. B **66**, 115305 (2002).
[4] H. Yi, Phys. Rev. B **65**, 195101 (2002).
[5] S. Lal, S. Rao, and D. Sen, Phys. Rev. B **66**, 165327 (2002).
[6] S. Chen, B. Trauzettel, and R. Egger, Phys. Rev. Lett. **89**, 226404 (2002).
[7] R. Egger, B. Trauzettel, S. Chen, and F. Siano, New J. Phys. **5**, 117 (2003).
[8] K.-V. Pham, F. Piechon, K.-I. Imura, and P. Lederer, Phys. Rev. B **68**, 205110 (2003).
[9] C. Chamon, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. **91**, 206403 (2003); M. Oshikawa, C. Chamon, and I. Affleck, J. Stat. Mech.: Theory Exp. 2006, P02008.
[10] S. Rao and D. Sen, Phys. Rev. B **70**, 195115 (2004).
[11] E. A. Kim, S. Vishveshwara, and E. Fradkin, Phys. Rev. Lett. **93**, 266803 (2004).
[12] E.-A. Kim, M. J. Lawler, S. Vishveshwara, and E. Fradkin, Phys. Rev. Lett. **95**, 176402 (2005); Phys. Rev. B **74**, 155324 (2006).
[13] A. Furusaki, J. Phys. Soc. Jpn. **74**, 73 (2005).
[14] D. Giuliano and P. Sodano, Nucl. Phys. B **711**, 480 (2005); D. Giuliano and P. Sodano, New J. Phys. **10**, 093023 (2008); D. Giuliano and P. Sodano, arXiv:0808.2678 (unpublished).
[15] T. Enss, V. Meden, S. Andergassen, X. Barnabe-Theriault, W. Metzner, and K. Schonhammer, Phys. Rev. B **71**, 155401 (2005); X. Barnabe-Theriault, A. Sedeki, V. Meden, and K. Schonhammer, Phys. Rev. Lett. **94**, 136405 (2005); X. Barnabe-Theriault, A. Sedeki, V. Meden, and K. Schonhammer, Phys. Rev. B **71**, 205327 (2005).
[16] K. Kazymyrenko and B. Douçot, Phys. Rev. B **71**, 075110 (2005).
[17] D. Friedan, arXiv:cond-mat/0505084 (unpublished); D. Friedan, arXiv:cond-mat/0505085 (unpublished).
[18] S. Das, S. Rao, and D. Sen, Phys. Rev. B **74**, 045322 (2006).
[19] H. Guo and S. R. White, Phys. Rev. B **74**, 060401(R) (2006).
[20] B. Bellazzini and M. Mintchev, J. Phys. A **39**, 11101 (2006).
[21] B. Bellazzini, M. Mintchev, and P. Sorba, J. Phys. A **40**, 2485 (2007).
[22] B. Bellazzini, M. Burrello, M. Mintchev, and P. Sorba, Proc. Symp. Pure Math. **77**, 639 (2007).
[23] C.-Y. Hou and C. Chamon, Phys. Rev. B **77**, 155422 (2008).
[24] S. Das, S. Rao, and A. Saha, Phys. Rev. B **77**, 155418 (2008); S. Das, S. Rao, and A. Saha, Europhys. Lett. **81**, 67001 (2008).
[25] S. Das and S. Rao, Phys. Rev. B **78**, 205421 (2008).
[26] C.-Y. Hou, E.-A. Kim, and C. Chamon, arXiv:0808.1723 (unpublished).
[27] The literature on bosonization and Luttinger liquids is enormous. we can just quote few general references here, i.e., A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, U.K., 2004); J. von Delft and H. Schoeller, Ann. Phys. **7**, 225 (1998); S. Rao, D. Sen, and F. D. M. Haldane, arXiv:cond-mat/0005492 (unpublished); S. Rao, D. Sen, and F. D. M. Haldane, J. Phys. C **15**, 2585 (1981).
[28] P. di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997); J. Cardy, *Encyclopedia of Mathematical Physics* (Elsevier, Amsterdam, 2006); J. Cardy, arXiv:0807.3472 (unpublished).

[29] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46**, 15233 (1992).
[30] A. Furusaki and N. Nagaosa, Phys. Rev. B **47**, 4631 (1993).
[31] E. Wong and I. Affleck, Nucl. Phys. B **417**, 403 (1994).
[32] G. Delfino, G. Mussardo, and P. Simonetti, Nucl. Phys. B **432**, 518 (1994).
[33] M. Oshikawa and I. Affleck, Nucl. Phys. B **495**, 533 (1997).
[34] A. Liguori and M. Mintchev, Nucl. Phys. B **522**, 345 (1998).
[35] H. Saleur, arXiv:cond-mat/9812110 (unpublished).
[36] R. Konik and A. LeClair, Nucl. Phys. B **538**, 587 (1999).
[37] A. LeClair and A. Ludwig, Nucl. Phys. B **549**, 546 (1999).
[38] A. Liguori, M. Mintchev, and L. Zhao, Commun. Math. Phys. **194**, 569 (1998); M. Mintchev, E. Ragoucy, and P. Sorba, Phys. Lett. B **547**, 313 (2002); M. Mintchev, E. Ragoucy, and P. Sorba, J. Phys. A **36**, 10407 (2003); M. Mintchev and P. Sorba, J. Stat. Mech.: Theory Exp. 2004, P07001.
[39] C. Bachas, J. de Boer, R. Dijkgraaf, and H. Ooguri, J. High Energy Phys. **2002**, P06027.
[40] M. Mintchev and P. Sorba, Ann. Henri Poincare **7**, 1375 (2006).
[41] J. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[42] A. Kundu, Phys. Rev. Lett. **83**, 1275 (1999).
[43] M. T. Batchelor, X. W. Guan, and N. Oelkers, Phys. Rev. Lett. **96**, 210402 (2006); M. T. Batchelor, X. W. Guan, and J. S. He, J. Stat. Mech.: Theory Exp. 2007, P03007; M. T. Batchelor and X. W. Guan, Phys. Rev. B **74**, 195121 (2006); M. T. Batchelor and X. W. Guan, Laser Phys. Lett. **4**, 77 (2007).
[44] L. Amico, A. Osterloh, and U. Eckern, Phys. Rev. B **58**, R1703 (1998); A. Osterloh, L. Amico, and U. Eckern, J. Phys. A **33**, L487 (2000); A. Osterloh, L. Amico, and U. Eckern, Nucl. Phys. B **588**, 531 (2000).
[45] M. D. Girardeau, Phys. Rev. Lett. **97**, 210401 (2006).
[46] R. Santachiara, F. Stauffer, and D. Cabra, J. Stat. Mech.: Theory Exp. 2007, L05003.
[47] J. X. Zhu and Z. D. Wang, Phys. Rev. A **53**, 600 (1996).
[48] P. Calabrese and M. Mintchev, Phys. Rev. B **75**, 233104 (2007).
[49] O. I. Patu, V. E. Korepin, and D. V. Averin, J. Phys. A **40**, 14963 (2007); O. I. Patu, V. E. Korepin, and D. V. Averin, *ibid.* **41**, 255205 (2008); O. I. Patu, V. E. Korepin, and D. V. Averin, *ibid.* **41**, 145006 (2008); arXiv:0811.2419 (unpublished).
[50] D. V. Averin and J. A. Nesteroff, Phys. Rev. Lett. **99**, 096801 (2007).
[51] A. Liguori, M. Mintchev, and L. Pilo, Nucl. Phys. B **569**, 577 (2000); A. Liguori and M. Mintchev, Commun. Math. Phys. **169**, 635 (1995).
[52] N. Ilieva and W. Thirring, Eur. Phys. J. C **6**, 705 (1999); N. Ilieva and W. Thirring, Theor. Math. Phys. **121**, 1294 (1999).
[53] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, Phys. Rev. Lett. **98**, 160409 (2007); S. Trebst, E. Ardonne, A. Feiguin, D. A. Huse, A. W. W. Ludwig, and M. Troyer, *ibid.* **101**, 050401 (2008); L. Fidkowski, G. Refael, N. Bonesteel, and J. Moore, arXiv:0807.1123 (unpublished).
[54] M. Greiter, arXiv:0707.1011 (unpublished).
[55] R.-G. Zhu and A.-M. Wang, arXiv:0712.1264 (unpublished).
[56] R. Santachiara and P. Calabrese, J. Stat. Mech.: Theory Exp. 2008, P06005; P. Calabrese and R. Santachiara, arXiv:0811.2991 (unpublished).
[57] Y. Hao, Y. Zhang, and S. Chen, Phys. Rev. A **78**, 023631 (2008).
[58] A. del Campo, Phys. Rev. A **78**, 045602 (2008).
[59] M. Batchelor, X.-W. Guan, and A. Kundu, J. Phys. A **41**, 352002 (2008).
[60] L. Jiang, G. K. Brennen, A. V. Gorshkov, K. Hammerer, M. Hafezi, E. Demler, M. D. Lukin, and P. Zoller, Nat. Phys. **4**, 482 (2008); B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **87**, 010402 (2001); M. Aguado, G. K. Brennen, F. Verstraete, and J. I. Cirac, *ibid.* **101**, 260501 (2008).
[61] P. Kuchment, arXiv:0802.3442 (unpublished).
[62] V. Kostrykin and R. Schrader, Fortschr. Phys. **48**, 703 (2000).
[63] M. Harmer, J. Phys. A **33**, 9015 (2000); M. Harmer, *ibid.* **33**, 9193 (2000).
[64] B. Bellazzini, M. Mintchev, and P. Sorba, arXiv:0810.3101 (unpublished).
[65] J. Cardy, Nucl. Phys. B **240**, 514 (1984).
[66] Excluding some exceptional boundary conditions in graphs with even number $n=2m$ of edges for which the system behaves as a bunch of m independent lines.
[67] I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991); I. Affleck and A. W. W. Ludwig, Phys. Rev. B **48**, 7297 (1993).
[68] D. Friedan and A. Konechny, Phys. Rev. Lett. **93**, 030402 (2004).
[69] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008); J. Cardy, Eur. Phys. J. B **64**, 1434 (2008); J. Eisert, M. Cramer, and M. B. Plenio, arXiv:0808.3773 (unpublished).
[70] I. Peschel, J. Phys. A **38**, 4327 (2005); J. Zhao, I. Peschel, and X. Wang, Phys. Rev. B **73**, 024417 (2006).
[71] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B **424**, 443 (1994); P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp. 2004, P06002; P. Calabrese and J. Cardy, Int. J. Quantum Inf. **4**, 429 (2006).
[72] P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp. 2007, P10004.
[73] I. Klich and L. Levitov, arXiv:0804.1377 (unpublished).
[74] R. Guyon, P. Devillard, T. Martin, and I. Safi, Phys. Rev. B **65**, 153304 (2002).